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OPTIMAL DESIGN OF VIBRATING CIRCULAR PLATES

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Abstract—Related to a given volume, the shape of a rotationally symmetric plate is determined so that its first natural frequency of transverse vibrations becomes optimal. Three different cases of boundary conditions are investigated.

Assuming that the lowest mode is rotationally symmetric, the corresponding mathematical problem is shown to be an ordinary, fourth order, non-linear and singular, but homogeneous eigenvalue problem. The differential equation is derived by variational analysis, and is solved numerically by successive iterations.

INTRODUCTION

THIS paper deals with the problem of optimizing a thin, circular plate with rotational symmetry, assuming the volume, diameter and Poisson's ratio of the plate to be given. The shape is determined from the condition that the lowest natural frequency of free, harmonic transverse vibrations of a rotationally symmetric mode is as high as possible.

It is assumed that the lowest natural frequency of a plate with rotational symmetry results from a rotationally symmetric mode. Then, the problem can be formulated as follows: Determine the thickness function so that the lowest natural frequency of transverse vibrations becomes a maximum, and determine the value of this frequency.

The problem is solved for the following three cases: a simply supported plate, a clamped plate, and a plate supported at the center with a free edge. Under these conditions the optimal shape, the corresponding lowest mode of transverse vibrations, and the first natural frequency is determined.

Related problems have been treated by Niordson [1] and Keller and Niordson [2]. The method applied here is similar to the method developed in [1].

The analysis of the problem is based upon the general theory of thin elastic plates, which yields a fourth order, linear, homogeneous, ordinary differential equation governing the motion of the plate in free, harmonic, transverse vibrations with rotationally symmetric modes.

Together with any of the boundary conditions mentioned above, this equation defines an eigenvalue problem which is full-definite and self-adjoint. The Rayleigh quotient associated with all three problems is shown to be the same.

Treating the Rayleigh quotient by variational analysis, a fourth order, homogeneous, non-linear differential equation of the eigenfunction is obtained. Thus, the mathematical problem is reduced to solve a non-linear eigenvalue problem, consisting of a non-linear differential equation together with the specific boundary conditions. A solution has been obtained numerically by successive iterations with the help of a digital computer.

It is shown that each solution is singular. The type of each singularity is determined analytically, and each singular function is separated into a simple function which includes the singularity, and a regular function. Only the regular function is subject to numerical treatment.

1. PRELIMINARY CONSIDERATIONS

For a given plate free, harmonic, transverse vibrations is governed by the equation

$$\{r[D(w''+w'/r)'+D'(w''+\nu w'/r)]\}' = \omega^2 \rho H w r, \tag{1.1}$$

where r is the radius,

$D = EH^3/12(1-\nu^2)$ is the flexural rigidity of the plate,
$H(r)$ is the thickness function,
E is Young's modulus,
ν is Poisson's ratio,
$w(r)$ is the lateral deflection,
ω is the natural angular frequency, and
ρ is the mass density.

This fourth order, homogeneous, linear, ordinary differential equation may for example be obtained from [3], p.299. The primes denote here differentiation with respect to the independent variable, r.

Because of rotational symmetry, there are only radial and tangential bending moments M_r, M_t, and a radial shearing force Q_r. These stress resultants (per unit length) are taken positive as indicated in Fig. 1. They can be expressed in terms of the deflection w as follows:

$$M_r = -D(w''+\nu w'/r) \tag{1.2}$$

$$M_t = -D(\nu w''+w'/r) \tag{1.3}$$

$$Q_r = -D(w''+w'/r)'-D'(w''+\nu w'/r). \tag{1.4}$$

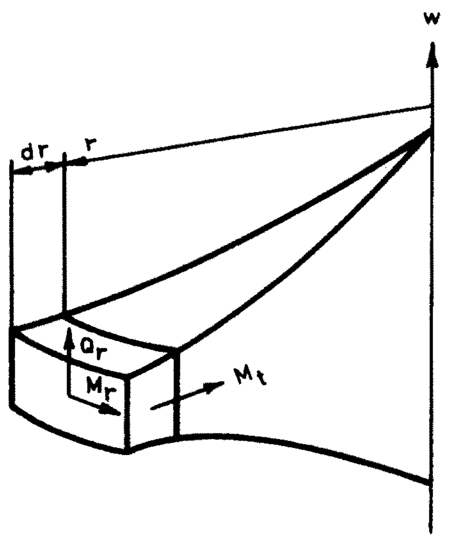

FIG. 1. Stress resultants.

The differential equation (1.1) is written in a non-dimensional form by introducing the dimensionless radial distance x,

$$x = r/R, \tag{1.5}$$

and the dimensionless plate thickness $h(x)$,

$$h = HR^2/V, \tag{1.6}$$

where R is the outer radius and where

$$V = \int_0^R 2\pi Hr \, dr \tag{1.7}$$

is the total volume of the plate.

Introducing x and h from (1.5) and (1.6) into (1.1) we obtain

$$\{x[h(x)^3(w''+w'/x)'+(h(x)^3)'(w''+\nu w'/x)]\}' = \lambda h(x)wx, \tag{1.8}$$

where the primes now denote differentiation with respect to the new independent variable x, and where the parameter λ is defined by

$$\lambda = 12(1-\nu^2)\omega^2\rho R^8/EV^2. \tag{1.9}$$

Since the differential equation (1.8) is linear and homogeneous we can regard w as dimensionless. Before establishing boundary conditions for the differential equation, we introduce the non-dimensional stress resultants m_r (radial bending moment) and q_r (radial shearing force),

$$m_r(x) = -h(x)^3(w''+\nu w'/x) \tag{1.10}$$

$$q_r(x) = -h(x)^3(w''+w'/x)'-(h(x)^3)'(w''+\nu w'/x). \tag{1.11}$$

Centrally supported plate

The mathematical problem governing transverse vibrations of a centrally supported plate with a free edge consists of the differential equation (1.8) and four boundary conditions. At the center of the plate ($x = 0$), the deflection w is equal to zero. Also, the first derivative w' must vanish at $x = 0$ due to the symmetry of w. At the free edge where $x = 1$, the moment m_r as well as the shearing force q_r must vanish. These conditions are expressed by

$$w(0) = 0 \tag{1.12a}$$

$$w'(0) = 0 \tag{1.12b}$$

$$\{h(x)^3(w''+\nu w'/x)\}_{x=1} = 0 \tag{1.12c}$$

$$\{h(x)^3(w''+w'/x)'+(h(x)^3)'(w''+\nu w'/x)\}_{x=1} = 0. \tag{1.12d}$$

Simply supported plate

When the plate is simply supported, the deflection w as well as the radial bending moment m_r must be zero at the supported edge. At the center of the plate, the first derivative w', and the radial shearing force q_r is clearly zero because of the symmetry. Thus, the boundary conditions become

$$w'(0) = 0 \tag{1.13a}$$

$$\{h(x)^3(w''+w'/x)'+(h(x)^3)'(w''+\nu w'/x)\}_{x=0} = 0 \tag{1.13b}$$

$$w(1) = 0 \tag{1.13c}$$

$$\{h(x)^3(w''+\nu w'/x)\}_{x=1} = 0. \tag{1.13d}$$

Clamped plate

In the case of a clamped plate, we have the following conditions. At the edge, the deflection w and its derivative w' must vanish. At the center, again from reasons of symmetry the derivative w' and the radial shearing force q_r must be zero. Hence,

$$w'(0) = 0 \tag{1.14a}$$

$$\{h(x)^3(w''+w'/x)'+(h(x)^3)'(w''+\nu w'/x)\}_{x=0} = 0 \tag{1.14b}$$

$$w(1) = 0 \tag{1.14c}$$

$$w'(1) = 0. \tag{1.14d}$$

The eigenvalue problems

Together with the differential equation (1.8), each set of the boundary conditions (1.12), (1.13) and (1.14) constitutes an eigenvalue problem. These problems will now be discussed briefly from a general point of view. Following the notation in [4], we introduce the shorter notations $M\{w\}$ and $N\{w\}$ for the differential operators, so that

$$M\{w\} = \{x[h(x)^3(w''+w'/x)'+(h(x)^3)'(w''+\nu w'/x)]\}' \tag{1.15}$$

and

$$N\{w\} = h(x)wx. \tag{1.16}$$

Hence equation (1.8) may be written in the form

$$M\{w\} = \lambda N\{w\}. \tag{1.17}$$

By partial integration and by considering the boundary conditions, it can be easily shown that the eigenvalue problem is self-adjoint, i.e. the relations

$$\int_0^1 (uM\{v\}-vM\{u\})\,dx = 0 \tag{1.18}$$

$$\int_0^1 (uN\{v\}-vN\{u\})\,dx = 0 \tag{1.19}$$

are satisfied for each pair of functions $u(x)$ and $v(x)$ which are sufficiently differentiable and satisfy all boundary conditions. Functions, which fulfill those conditions and which do not vanish identically, are called comparison functions.

Furthermore, if $u(x)$ is any comparison function, we find that

$$\int_0^1 uM\{u\}\,dx = \int_0^1 h(x)^3\{(1-\nu)[(u'')^2+(u'/x)^2]+\nu[u''+u'/x]^2\}x\,dx > 0 \tag{1.20}$$

and

$$\int_0^1 uN\{u\}\,dx = \int_0^1 h(x)u^2x\,dx > 0, \tag{1.21}$$

since the plate thickness function $h(x)$ is non-negative. Hence the problem is full-definite.

The Rayleigh quotient $R\{u\}$ is defined as the functional [4]

$$R\{u\} = \frac{\int_a^b uM\{u\}\,dx}{\int_a^b uN\{u\}\,dx}, \tag{1.22}$$

a and b being the end points of the interval. Re-writing equation (1.20) and substituting (1.20) and (1.21) into (1.22), we obtain

$$R\{u\} = \frac{\int_0^1 h(x)^3\{(u'')^2+2\nu u''u'/x+(u'/x)^2\}x\,dx}{\int_0^1 h(x)u^2x\,dx} \tag{1.23}$$

an expression, which holds in all three cases. In fact, this is the so-called Kamke quotient corresponding to the problems, and we assume that the Rayleigh minimum principle

$$\lambda_1 \leq R\{u\} \tag{1.24}$$

where λ_1 is the lowest eigenvalue, is valid, provided that the function $u(x)$ is *admissible*, i.e. satisfies the essential (kinematical) boundary conditions of the problem, and is sufficiently differentiable.

2. VARIATIONAL ANALYSIS

In this section, we restrict our attention to the lowest eigenvalue λ_1. It is evident that the Rayleigh quotient attains the eigenvalue λ_1, when the eigenfunction $w(x)$ is substituted for u. Hence,

$$\lambda_1 = R\{w\} = \frac{\int_0^1 h(x)^3\{(w'')^2+2\nu w''w'/x+(w'/x)^2\}x\,dx}{\int_0^1 h(x)w^2x\,dx}. \tag{2.1}$$

Let us assume that the volume of the plate is given. Then the thickness function $h(x)$ is subject to the condition

$$\int_0^1 2\pi h(x)x\,dx = 1, \tag{2.2}$$

cf. (1.5)–(1.7). Now we ask if we can determine a function h subject to the restriction (2.2), which optimizes λ_1. In order to answer this question, we shall make a variational analysis of the relation (2.1) which is valid in all three cases. The equations derived in this analysis are the same in all three cases.

Following [1], we assume that an optimal thickness function $h(x)$ and a corresponding eigenfunction $w(x)$ exist, and we consider a family of functions $h(x, \varepsilon)$, containing the solution $h(x, 0)$. We shall assume that all members of this family have the following properties. The functions are differentiable with respect to the parameter ε, they are non-negative, and they satisfy the condition (2.2). Associated to every function $h(x, \varepsilon)$ is a first eigenfunction $w(x, \varepsilon)$. These functions $w(x, \varepsilon)$ are the solutions of eigenvalue problems with the thickness functions $h(x, \varepsilon)$, and it follows from a well known theorem for differential equations that also the solutions $w(x, \varepsilon)$ will be differentiable with respect to ε. Also, all functions $w(x, \varepsilon)$ will be admissible functions for the eigenvalue problem corresponding to the optimal plate thickness function $h(x, 0)$.

Having the expression (2.1) in mind, we now define a functional R, depending only upon the parameters ε and δ, by the equation

$$R(\varepsilon, \delta) = \frac{\int_0^1 h(x, \varepsilon)^3\{[w''(x, \delta)]^2 + 2\nu w''(x, \delta)w'(x, \delta)/x + [w'(x, \delta)/x]^2\}x\,\mathrm{d}x}{\int_0^1 h(x, \varepsilon)[w(x, \delta)]^2 x\,\mathrm{d}x}. \quad (2.3)$$

It is evident now that

$$R(\varepsilon, \varepsilon) = \lambda_1(\varepsilon) \quad (2.4)$$

is the eigenvalue related to the function $h(x, \varepsilon)$, when $\varepsilon = \delta$. In particular, when $\varepsilon = \delta = 0$, then $R(0, 0) = \lambda_1(0)$. It is this value which should be determined, and since it is an optimal value, we have

$$(\mathrm{d}\lambda_1/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0} = 0. \quad (2.5)$$

Since $h(x, \varepsilon)$ and $w(x, \varepsilon)$ both are differentiable with respect to ε, the function $R(\varepsilon, \delta)$ also has to be that at the point $(\varepsilon, \delta) = (0, 0)$. Hence, we obtain from (2.4) and (2.5)

$$(\mathrm{d}R/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0} = (\partial R/\partial\varepsilon)_{\varepsilon=\delta=0} + (\partial R/\partial\delta)_{\varepsilon=\delta=0}(\mathrm{d}\delta/\mathrm{d}\varepsilon)_{\varepsilon=\delta=0} = 0. \quad (2.6)$$

In fact, when $\varepsilon = 0$, the expression on the right hand side of equation (2.3) is the Rayleigh quotient corresponding to the optimal thickness function $h(x, 0)$. As outlined at the end of the previous section, this quotient will be minimized by the eigenfunction related to the problem, i.e. by $w(x, 0)$. We may therefore write

$$(\partial R/\partial\delta)_{\varepsilon=\delta=0} = 0 \quad (2.7)$$

from which we obtain the following necessary condition for a solution $h(x, 0)$:

$$(\partial R/\partial\varepsilon)_{\varepsilon=\delta=0} = 0. \quad (2.8)$$

Applying this condition to (2.3), and denoting the optimal thickness function and the corresponding lowest eigenfunction simply by $h(x)$ and $w(x)$ respectively, we obtain

$$\begin{aligned} &\int_0^1 hw^2x\,\mathrm{d}x \int_0^1 3h^2(\partial h/\partial\varepsilon)_{\varepsilon=0}[(w'')^2 + 2\nu w''w'/x + (w'/x)^2]x\,\mathrm{d}x \\ &\quad - \int_0^1 h^3[(w'')^2 + 2\nu w''w'/x + (w'/x)^2]x\,\mathrm{d}x \int_0^1 (\partial h/\partial\varepsilon)_{\varepsilon=0}w^2x\,\mathrm{d}x = 0. \end{aligned} \quad (2.9)$$

Dividing through by $\int_0^1 hw^2 x\,dx$ and taking equation (2.1) into account, we find the equation

$$\int_0^1 \{3h^2[(w'')^2+2\nu w''w'/x+(w'/x)^2]-\lambda_1 w^2\}(\partial h/\partial\varepsilon)_{\varepsilon=0}x\,dx = 0. \tag{2.10}$$

From the condition (2.2) which has to be satisfied also by the optimal thickness function, we get

$$\int_0^1 (\partial h/\partial\varepsilon)_{\varepsilon=0}x\,dx = 0. \tag{2.11}$$

However, both equations (2.10) and (2.11) cannot be satisfied unless the function within the brackets { } of (2.10) is independent of x, i.e. constant. Thus, we have

$$3h^2[(w'')^2+2\nu w''w'/x+(w'/x)^2]-\lambda_1 w^2 = \lambda_1 A, \tag{2.12}$$

where $\lambda_1 A$ is a constant yet to be determined. Thus, we have the following equation for a possible optimal thickness function expressed in terms of its corresponding eigenfunction:

$$h(x) = (\lambda_1/3)^{\frac{1}{2}}\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}}. \tag{2.13}$$

In order to obtain the quantity A and the eigenvalue λ_1, we multiply equation (2.12) by hx and integrate over the interval. We find

$$3\int_0^1 h^3[(w'')^2+2\nu w''w'/x+(w'/x)^2]x\,dx-\lambda_1\int_0^1 hw^2x\,dx = \lambda_1 A\int_0^1 hx\,dx. \tag{2.14}$$

If we substitute (2.1) into this equation, and use the condition (2.2) together with the equation (2.13) for $h(x)$, we obtain

$$A = 4\pi(\lambda_1/3)^{\frac{1}{2}}\int_0^1\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}} w^2x\,dx. \tag{2.15}$$

On the other hand, combining (2.13) with (2.2) yields the relation

$$\lambda_1 = \frac{3}{4\pi^2\left\{\int_0^1\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}}x\,dx\right\}^2}. \tag{2.16}$$

Substitution of this expression into (2.15) results in the following implicit equation for the quantity A

$$A = 2\frac{\int_0^1\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}}w^2x\,dx}{\int_0^1\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}}x\,dx}. \tag{2.17}$$

Finally, substituting (2.13) into the differential equation (1.8) which governs the function w, we obtain the following ordinary, fourth order, non-linear differential equation for the eigenfunction w of the optimal plate:

$$\left\{x\left[\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{3}{2}}(w''+w'/x)' + \left[\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{3}{2}}\right]'(w''+\nu w'/x)\right]\right\}' = 3\left[\frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2}\right]^{\frac{1}{2}} wx. \tag{2.18}$$

Now we are able to conclude, that in each case, i.e. for each set of boundary conditions we must investigate, whether the principal differential equation (2.18) in which the constant A must satisfy the relation (2.17) has a solution which satisfies corresponding boundary conditions. If so, our task is to determine this solution. As already anticipated, solutions have been obtained, and from the function w and the constant A of these solutions the thickness function $h(x)$ and the eigenvalue λ_1 is uniquely determined by the equations (2.13) and (2.16).

3. CENTRALLY SUPPORTED PLATE

Formulation of the mathematical problem

Introducing the shorter notation $T(x)$ for the function

$$T(x) = \frac{A+w^2}{(w'')^2+2\nu w''w'/x+(w'/x)^2} \tag{3.1}$$

which according to (2.13) is proportional to the square of the thickness function of the optimal plate, the principal differential equation (2.18) of the eigenfunction can be written

$$\{x[T(x)^{\frac{3}{2}}(w''+w'/x)'+(T(x)^{\frac{3}{2}})'(w''+\nu w'/x)]\}' = 3T(x)^{\frac{1}{2}}wx. \tag{3.2}$$

The quantity A must satisfy the implicit relation (2.17), which is now written in the form

$$A = 2\frac{\int_0^1 T(x)^{\frac{1}{2}}w^2x\,dx}{\int_0^1 T(x)^{\frac{1}{2}}x\,dx}. \tag{3.3}$$

The boundary conditions of the problem are obtained from equations (1.12), when taking (2.13) and (3.1) into account:

$$w(0) = 0 \tag{3.4a}$$

$$w'(0) = 0 \tag{3.4b}$$

$$\{T(x)^{\frac{3}{2}}(w''+\nu w'/x)\}_{x=1} = 0 \tag{3.4c}$$

$$\{T(x)^{\frac{3}{2}}(w''+w'/x)'+(T(x)^{\frac{3}{2}})'(w''+\nu w'/x)\}_{x=1} = 0. \tag{3.4d}$$

It should be noted that the problem is a fourth order, non-linear, but homogeneous eigenvalue problem, where w is the eigenfunction and A is the eigenvalue.

Method of solution

The problem can be solved numerically by successive iterations, and the method is based upon a formal integration of the differential equation, which includes four steps of integration. Before each of these steps, the differential operator of the highest order is separated on the left hand side. If the equation contains more terms of the highest order, as will actually occur, then we separate the highest power of the highest order term. It was found that this was necessary in order to obtain convergence by successive iterations. Furthermore, at each integration one of the boundary conditions is introduced.

Consequently, we start the integration of (3.2) by applying condition (3.4d), and obtain

$$T(x)^{\frac{3}{2}}(w''+w'/x)'+(T(x)^{\frac{3}{2}})'(w''+\nu w'/x) = -\frac{3}{x}\int_x^1 T(x)^{\frac{1}{2}}wx\,\mathrm{d}x. \tag{3.5}$$

Now, we add the term $T(x)^{\frac{3}{2}}(w''+\nu w'/x)'$ and write the equation in the form

$$\begin{aligned}(T(x)^{\frac{3}{2}})'(w''+\nu w'/x)+T(x)^{\frac{3}{2}}(w''+\nu w'/x)'\\ = -\frac{3}{x}\int_x^1 T(x)^{\frac{1}{2}}wx\,\mathrm{d}x-T(x)^{\frac{3}{2}}[(w''+w'/x)'-(w''+\nu w'/x)'].\end{aligned} \tag{3.6}$$

The function on the left hand side is now the derivative of the function $T(x)^{\frac{3}{2}}(w''+\nu w'/x)$, and condition (3.4c) imposes the value zero for this at $x = 1$. On the right hand side, the two third-order terms vanish, and the operator on this side is of lower order now. Integrating, we then find

$$T(x)^{\frac{3}{2}}(w''+\nu w'/x) = \int_x^1 \left\{\frac{3}{x}\int_x^1 T(x)^{\frac{1}{2}}wx\,\mathrm{d}x+\frac{1-\nu}{x}T(x)^{\frac{3}{2}}(w''-w'/x)\right\}\mathrm{d}x. \tag{3.7}$$

Solving this equation with respect to the second-order term of the highest power, the expression (3.1) for $T(x)$ being substituted, gives us

$$w''(x) = \left[(A+w^2)\left\{\frac{w''+\nu w'/x}{\mathrm{m}(x)}\right\}^{\frac{2}{3}}-2\nu w''w'/x-(w'/x)^2\right]^{\frac{1}{2}} \tag{3.8}$$

where

$$m(x) = \int_x^1 \left\{\frac{3}{x}\int_x^1 T(x)^{\frac{1}{2}}wx\,\mathrm{d}x+\frac{1-\nu}{x}T(x)^{\frac{3}{2}}(w''-w'/x)\right\}\mathrm{d}x. \tag{3.9}$$

Integrating twice and using the conditions (3.4a) and (3.4b), we get

$$w(x) = \int_0^x\int_0^x\left[(A+w^2)\left\{\frac{w''+\nu w'/x}{m(x)}\right\}^{\frac{2}{3}}-2\nu w''w'/x-(w'/x)^2\right]^{\frac{1}{2}}\mathrm{d}x^2. \tag{3.10}$$

Now the following basic procedure for successive iterations was applied for solving the problem:

(1) $w_n'(x) = \int_0^x w_n''(x)\,\mathrm{d}x$

(2) $w_n(x) = \int_0^x w_n'(x)\,\mathrm{d}x$

(3) Determination of the constant A_n from the implicit equation

$$A_n = 2\frac{\int_0^1 T_n(x)^{\frac{1}{2}}(w_n)^2 x\,dx}{\int_0^1 T_n(x)^{\frac{1}{2}} x\,dx}$$

where

$$T_n(x) = \frac{A_n + (w_n)^2}{(w_n'')^2 + 2\nu w_n'' w_n'/x + (w_n'/x)^2}$$

The constant A_n is determined from this system of equations again by successive iterations (in an "inner loop"). When A_n is found, $T_n(x)$ is also determined.

(4) $$m_n(x) = \int_x^1 \left\{ \frac{3}{x}\int_x^1 T_n(x)^{\frac{1}{2}} w_n x\,dx + \frac{1-\nu}{x} T_n(x)^{\frac{3}{2}}(w_n'' - w_n'/x) \right\} dx$$

(5) $$w_{n+1}''(x) = \left[(A_n + (w_n)^2)\left\{ \frac{w_n'' + \nu w_n'/x}{m_n(x)} \right\}^{\frac{2}{3}} - 2\nu w_n'' w_n'/x - (w_n'/x)^2 \right]^{\frac{1}{2}}.$$

In this scheme, the subscript n is the iteration number. At the last step, the starting function for the next iteration w_{n+1}'' has been obtained.

Although the scheme indicates the general principle of the iteration procedure, the formulas cannot be applied directly as indicated. The reason for this is the singularities at the end points of the interval. Because of these singularities, the integrands found in the fourth step (4), and the integrands in the first two steps (1) and (2) of the scheme are not suitable for direct numerical integration in the vicinity of $x = 0$ and $x = 1$, respectively. In the following, we shall see how these difficulties were avoided.

At first, let us investigate the singularity at $x = 0$ analytically. We shall assume that the solution $w(x)$ can be expanded in the following series

$$w(x) = bx^2(-\log x)^{-p} + c_3x^3 + c_4x^4 + \dots \tag{3.11}$$

in the vicinity of $x = 0$. Here p is assumed to be a positive number less than 1, and the coefficient b to be different from zero.

Substituting (3.11) into the differential equation (3.2), and making the sum of coefficients of the leading terms equal to zero, p is found to be equal to 1/2. Hence, in the vicinity of $x = 0$ we have

$$w(x) = bx^2(-\log x)^{-\frac{1}{2}} + c_3x^3 + c_4x^4 + \dots \tag{3.12}$$

This solution clearly satisfies the boundary conditions (3.4a) and (3.4b). Furthermore, we find that $w''(0) = 0$, whereas the third derivative $w'''(x)$ tends to infinity as x tends to zero. Substituting (3.12) into (2.13), (1.10) and (1.11), we obtain

$$h(x) = \frac{1}{b}\left[\frac{\lambda_1 A}{24(1+\nu)}\right]^{\frac{1}{3}}(-\log x)^{\frac{1}{2}} \tag{3.13}$$

$$m_r(x) = \frac{2(1+\nu)}{b^2}\left[\frac{\lambda_1 A}{24(1+\nu)}\right]^{\frac{2}{3}} \log x \tag{3.14}$$

$$q_r(x) = \frac{2+3\nu}{b^2}\left[\frac{\lambda_1 A}{24(1+\nu)}\right]^{\frac{2}{3}} x^{-1}, \tag{3.15}$$

i.e. the thickness function h, the bending moment m_r, and the shearing force q_r of the optimal plate tend to infinity as x tends to zero.

It should be noted that the type of the singularity (3.15) of q_r may be checked as follows. Denoting the (concentrated) support reaction by P, the relation

$$2\pi x q_r(x) = P \tag{3.16}$$

must hold in the vicinity of $x = 0$ from reasons of equilibrium, since the inertia forces of the plate become negligible in comparison with P. Hence, from equation (3.16) q_r is again found to be proportional to x^{-1}.

In order to investigate the solution $w(x)$ in the vicinity of $x = 1$, we assume that $w(x)$ can be expanded in a power series of the type

$$w(x) = a(1-x)^p + \ldots + c_0 + c_1(1-x) + c_2(1-x)^2 + \ldots. \tag{3.17}$$

Now p is a real number, assumed to be the lowest negative power of the series for which the coefficient a does not vanish.

Substituting the series (3.17) into the differential equation (3.2) and equating to zero the sum of coefficients from lowest power terms, we obtain $p = -1$. Thus, in the vicinity of $x = 1$, we may expand $w(x)$ as follows

$$w(x) = a(1-x)^{-1} + \ldots + c_0 + c_1(1-x) + c_2(1-x)^2 + \ldots \tag{3.18}$$

Clearly, w and its derivatives tend to infinity as x tends to one. It should be noted however, that the solution satisfies the boundary conditions (3.4c) and (3.4d), since

$$\lim_{x\to 1} \{T(x)^{\frac{3}{2}}(w'' + \nu w'/x)\} = \lim_{x\to 1} \left\{\frac{a}{4}(1-x)^3\right\} = 0 \tag{3.19}$$

and

$$\lim_{x\to 1} \{T(x)^{\frac{3}{2}}(w'' + w'/x)' + (T(x)^{\frac{3}{2}})'(w'' + \nu w'/x)\} = \lim_{x\to 1} \left\{\frac{3a}{4}(1-x)^2\right\} = 0. \tag{3.20}$$

Furthermore, substituting (3.18) into (2.13), it appears that the thickness function of the optimal plate is proportional to the square of the distance from the edge.

The expansion formula (3.18) indicates how to define three finite functions $l(x)$, $k(x)$ and $g(x)$ in the closed interval $0 \leqq x \leqq 1$, by the following equations

$$w(x) = l(x)(1-x)^{-1} \tag{3.21}$$

$$w'(x) = k(x)(1-x)^{-2} \tag{3.22}$$

$$w''(x) = g(x)(1-x)^{-3}. \tag{3.23}$$

Furthermore, we define the functions $T^*(x)$ and $m^*(x)$ by the equations

$$T^*(x) = \frac{A(1-x)^2 + l(x)^2}{g(x)^2 + 2\nu(1-x)g(x)k(x)/x + \{(1-x)k(x)/x\}^2} \tag{3.24}$$

and

$$m^*(x) = T^*(x)^{\frac{3}{2}}\{g(x) - \nu(1-x)k(x)/x\}. \tag{3.25}$$

The functions $l(x)$, $k(x)$, $g(x)$, $T^*(x)$ and $m^*(x)$ now introduced, are regular in the interval $0 < x \leq 1$. At the point $x = 0$, the types of the singularities of these functions are equal to those of the functions $w(x)$, $w'(x)$, $w''(x)$, $T(x)$ and $m(x)$, respectively. Also, we have $l(0) = k(0) = g(0) = 0$, since w, w' and w'' vanish at $x = 0$.

Now, substituting these new functions, the scheme for successive iterations is transformed in the following one where finite functions only, are subject to numerical treatment in the integration steps:

(1) $k_n(x) = (1-x)^2 \int_0^x g_n(x)(1-x)^{-3}\,dx$

(2) $l_n(x) = (1-x) \int_0^x k_n(x)(1-x)^{-2}\,dx$

(3) Determination of the constant A_n from the implicit equation

$$A_n = 2\frac{\int_0^1 T_n^*(x)^{\frac{1}{2}}(l_n)^2 x\,dx}{\int_0^1 T_n^*(x)^{\frac{1}{2}}(1-x)^2 x\,dx}.$$

where

$$T_n^*(x) = \frac{A_n(1-x)^2 + l_n(x)^2}{g_n(x)^2 + 2\nu(1-x)g_n(x)k_n(x)/x + \{(1-x)k_n(x)/x\}^2}$$

Again, the constant A_n is determined from these equations by an inner loop of successive iterations.

(4) $$m_n^*(x) = (1-x)^{-3}\int_x^1 \left[3(1-x)^{-2}\int_x^1 T_n^*(x)^{\frac{1}{2}} l_n x(1-x)\,dx \right.$$
$$\left. +(1-\nu)(1-x)T_n^*(x)^{\frac{3}{2}}\{g_n - (1-x)k_n/x\}\right]\frac{(1-x)^2}{x}\,dx$$

(5) $$g_{n+1}(x) = \left[\{A_n(1-x)^2 + (l_n)^2\}\left\{\frac{g_n + \nu(1-x)k_n/x}{m_n^*}\right\}^{\frac{2}{3}}\right.$$
$$\left. -2\nu(1-x)g_n k_n/x - \{(1-x)k_n/x\}^2\right]^{\frac{1}{2}}.$$

The singular integrands in the first two formulas of this scheme are now represented as products of a simple function which takes care of the singularity at $x = 1$, and a finite function. The essential point is that the finite functions $g(x)$ and $k(x)$ are suitable for numerical integration.

In the formula contained in the fourth step of the scheme, the integrands are also separated, and the computation of the function m^* may be carried out numerically as outlined. In this formula, there are two reasons for the separation. At first, the integrals as well as the integrands vanish at the point $x = 1$. Hence the values of the integrals computed numerically in the vicinity of $x = 1$ would be affected by unacceptable relative errors, unless the functions $(1-x)$ and $(1-x)^2$ were separated. It may be noticed that the value $m^*(1)$ is different from zero, so that substitution into the last formula will cause no difficulties. Secondly, by separating the function x^{-1} in order to take care of the singularity at $x = 0$, we achieve that the function in the brackets [ ] becomes a finite function, appropriate for numerical integration.

The sequence of successive iterations may be started with an arbitrary regular function $g_0(x)$ which vanishes at the end point $x = 0$ of the interval. The sequence of iterates g_n, k_n, l_n, T_n^*, m_n^* converges rapidly to the functions g, k, l, T^*, m^*, from which the solution $w(x)$ and its derivatives are determined using equations (3.21)–(3.23). According to the equations (2.16) and (3.21)–(3.24), the optimal eigenvalue λ_1 is then found from the equation

$$\lambda_1 = \frac{3}{4\pi^2\left[\int_0^1 T^*(x)^{\frac{1}{2}}(1-x)^2 x\,\mathrm{d}x\right]^2}. \tag{3.26}$$

Using the relation

$$h(x) = (\lambda_1/3)^{\frac{1}{2}} T^*(x)^{\frac{1}{2}}(1-x)^2 \tag{3.27}$$

derived from (2.13) and (3.21)–(3.24), we may finally compute the thickness function $h(x)$ of the optimal plate.

The solution of the problem is illustrated in Fig. 2. The lower part shows a diametrical section through the plate. The corresponding eigenfunction and its first and second derivative are indicated above as functions of the dimensionless radial distance, x.

Numerical integration was performed by subdividing the interval $0 \leq x \leq 1$ into a number of equal parts and by applying a polynomial formula. The mesh length d was chosen to be 6/60, 5/60, 4/60, 3/60, 2/60, 1/60, and the result was extrapolated to $d = 0$ by means of Newton's formula. For the centrally supported plate with a free edge, the

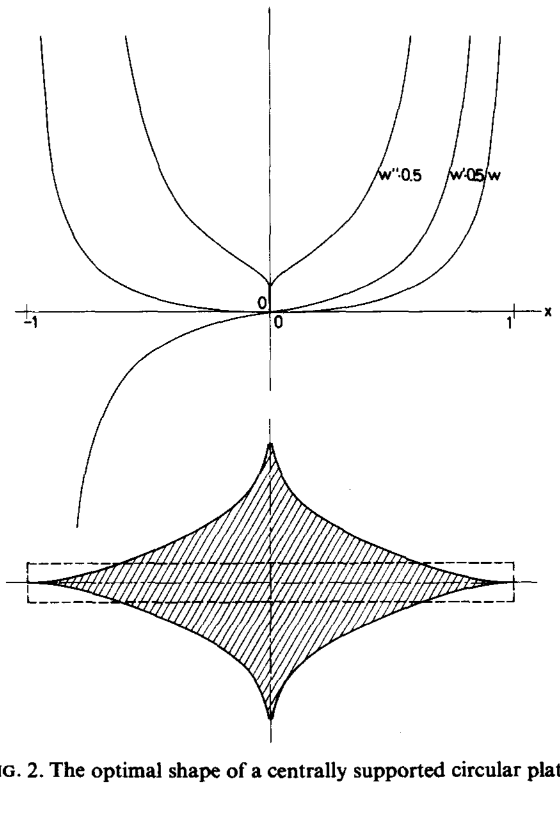

FIG. 2. The optimal shape of a centrally supported circular plate.

lowest eigenvalue λ_1 was found to be

$$\lambda_1 = 12(1-\nu^2)\omega_1^2 \rho R^8/EV^2 = 59{\cdot}27. \tag{3.28}$$

In obtaining this result, Poisson's ratio ν was assumed to be 0·3. It should be noted that the differential equation and the boundary conditions contain ν, which will actually influence the shape of the optimal plate, whereas Young's modulus E and the mass density ρ do not affect the shape.

By the dotted lines in Fig. 2 the shape of a centrally supported, circular plate with a free edge, having the same diameter, volume and material as the optimal plate, is indicated. In comparison with this plate of uniform thickness, the first natural frequency ω_1 of transverse vibrations of the optimal plate is increased with about 544 pct. (assuming $\nu = 0{\cdot}3$).

4. SIMPLY SUPPORTED PLATE

The boundary conditions of this problem are obtained by substituting the expression for the thickness function (2.13) into the conditions (1.13). Hence, using (3.1) we get

$$w'(0) = 0 \tag{4.1a}$$

$$\{T(x)^{\frac{3}{2}}(w''+w'/x)' + (T(x)^{\frac{3}{2}})'(w''+\nu w'/x)\}_{x=0} = 0 \tag{4.1b}$$

$$w(1) = 0 \tag{4.1c}$$

$$\{T(x)^{\frac{3}{2}}(w''+\nu w'/x)\}_{x=1} = 0. \tag{4.1d}$$

By integrating equation (3.2), we obtain the following expressions, where the boundary conditions have been used:

$$w(x) = -\int_x^1 \int_0^x \left[(A+w^2)\left\{\frac{w''+\nu w'/x}{m(x)}\right\}^{\frac{2}{3}} - 2\nu w''w'/x - (w'/x)^2\right]^{\frac{1}{2}} \mathrm{d}x^2 \tag{4.2}$$

$$m(x) = -\int_x^1 \left\{\frac{3}{x}\int_0^x T(x)^{\frac{1}{2}} wx\,\mathrm{d}x - \frac{1-\nu}{x} T(x)^{\frac{3}{2}}(w''-w'/x)\right\} \mathrm{d}x. \tag{4.3}$$

The solution $w(x)$ is also singular at $x = 1$. In the vicinity of this point we may expand the solution in the series

$$w(x) = c_1(1-x) + c_2(1-x)^2 + \ldots + a(1-x)^{\frac{3}{2}} + \ldots, \tag{4.4}$$

where 3/2 is the lowest non-integer power of the expansion. This solution (4.4) satisfies the boundary conditions (4.1c) and (4.1d), since

$$\lim_{x\to 1}\{w(x)\} = \lim_{x\to 1}\{c_1(1-x)\} = 0 \tag{4.5}$$

and

$$\lim_{x\to 1}\{T(x)^{\frac{3}{2}}(w''+\nu w'/x)\} = \lim_{x\to 1}\{(4/3a)^{\frac{1}{2}} A^{\frac{3}{2}}(1-x)\} = 0. \tag{4.6}$$

Substituting (4.4) into (2.13), we find that the thickness function in the vicinity of the edge is proportional to the square root of the distance from the edge.

The method applied for solving the problem (3.1)–(3.3) and (4.1a)–(4.1d) is very similar to the method described in the previous section. Thus, from the expressions (4.2) and (4.3) a fundamental procedure for successive iterations is constructed, which—because of the singularity of the solution $w(x)$—is transformed into an iteration scheme with separated integrands. In the present case however, there is no necessity of separating the function $w'(x)$, since it is finite in the entire interval, and can be satisfactorily represented numerically.

Figure 3 shows the solution of the problem. The thickness function of the optimal plate is shown below in a diametrical section of the plate, and the corresponding eigenfunction $w(x)$ and its first and second derivative are illustrated above. Notice that the derivative w'' tends to infinity at the edge, while w' is finite, in accordance with the expansion (4.4).

The eigenvalue λ_1 related to the optimal solution was found to be

$$\lambda_1 = 12(1-\nu^2)\omega_1^2\rho R^8/EV^2 = 3{\cdot}301, \tag{4.7}$$

when $\nu = 0{\cdot}3$. Compared with the natural frequency ω_1 of the corresponding uniform plate of equal volume and diameter indicated by the dotted lines on Fig. 3, the frequency of the optimal plate is increased with about 16 pct.

5. CLAMPED PLATE

The boundary conditions for a clamped plate are $w(1) = 0$, $w'(1) = 0$, $w'(0) = 0$ and $q_r(0) = 0$, where $q_r(x)$ is given by (1.11). If we substitute the expression (2.13) for the thickness function into the last equation, then the boundary conditions for our optimum problem are obtained.

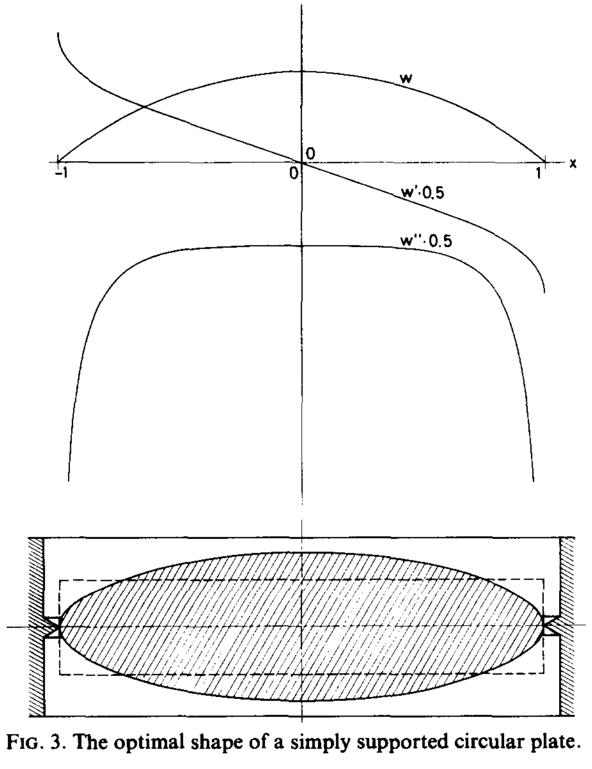

FIG. 3. The optimal shape of a simply supported circular plate.

The non-linear eigenvalue problem formulated in this way was carefully studied, and it appears that the problem contains a free parameter. Thus, corresponding to every arbitrarily chosen value (within certain limits) of this parameter, it is possible to find a solution of the problem. All these solutions are local optima however,† and it was concluded that the formulation given by the boundary conditions above, is not an adequate formulation of the problem of finding the optimal clamped plate.

As a consequence of this, the mathematical formulation was changed. The new formulation differs from the original one by one single boundary condition, namely by claiming the bending moment m_r of the optimal plate to be zero at the center, instead of the condition $w'(0) = 0$.

Consequently, we shall restrict our attention in the following to the solution of the equations (3.1)–(3.3) with the boundary conditions $w(1) = 0$, $w'(1) = 0$, $m_r(0) = 0$ and $q_r(0) = 0$. It should be pointed out that this new set of boundary conditions together with the differential equation (1.8) constitutes an eigenvalue problem, which again satisfies the relations (1.18)–(1.23). The basic assumptions for the variational analysis are then satisfied, and the result of this analysis is still valid.

Now, from (1.10), (1.11), (2.13) and (3.1) we find that the boundary conditions of our problem can be written

$$\{T(x)^{\frac{3}{2}}(w''+\nu w'/x)\}_{x=0} = 0 \tag{5.1a}$$

$$\{T(x)^{\frac{3}{2}}(w''+w'/x)'+(T(x)^{\frac{3}{2}})'(w''+\nu w'/x)\}_{x=0} = 0. \tag{5.1b}$$

$$w(1) = 0 \tag{5.1c}$$

$$w'(1) = 0. \tag{5.1d}$$

Considering these conditions, a formal integration of the equation (3.2) yields

$$w(x) = \int_x^1 \int_x^1 \left[(A+w^2)\left\{\frac{w''+\nu w'/x}{m(x)}\right\}^{\frac{2}{3}} - 2\nu w''w'/x-(w'/x)^2\right]^{\frac{1}{2}} \mathrm{d}x^2, \tag{5.2}$$

where

$$m(x) = \int_0^x x^{-1}\left\{3\int_0^x T(x)^{\frac{1}{2}}wx\,\mathrm{d}x-(1-\nu)T(x)^{\frac{3}{2}}(w''-w'/x)\right\}\mathrm{d}x. \tag{5.3}$$

From equations (5.2) and (5.3) a basic procedure for successive iterations is again constructed, but since the solution $w(x)$ of the problem now tends to infinity as x tends to zero, we separate this singularity and solve the problem similarly to the method described in Section 3.

In order to investigate the singularity, we assume that the solution in the vicinity of the point $x = 0$ can be expanded in a power series of the form

$$w(x) = ax^{-p}+\ldots+c_0+c_1x+c_2x^2+\ldots \tag{5.4}$$

with $a \neq 0$, and where $-p$ is the lowest negative power of the series. Substituting (5.4) into (3.2), and equating the sum of the coefficients of the leading terms to zero, we obtain the following equation for p:

$$p^3+(7-3\nu)p^2-2(1+3\nu)p-12(1-\nu) = 0. \tag{5.5}$$

† Depending on the choice of the free parameter.

For all possible values of the Poisson's ratio $0 \leq \nu \leq \frac{1}{2}$, this equation has three real roots, two of these being negative. Since p is assumed to be positive, the expansion (5.4) must hold, and when ν is given, the corresponding value of p is uniquely determined as the only positive root of (5.5).

The solution (5.4) satisfies the boundary conditions (5.1a) and (5.1b), since the relations

$$\lim_{x\to 0}\{T(x)^{\frac{3}{2}}(w''+\nu w'/x)\} = \lim_{x\to 0}\{k_1 x^{4-p}\} = 0 \tag{5.6}$$

and

$$\lim_{x\to 0}\{T(x)^{\frac{3}{2}}(w''+w'/x)'+(T(x)^{\frac{3}{2}})'(w''+\nu w'/x)\} = \lim_{x\to 0}\{k_2 x^{3-p}\} = 0 \tag{5.7}$$

where k_1 and k_2 are constants, are satisfied for all relevant values of ν. For instance, corresponding to $\nu = 0{\cdot}3$ we find $p = 1{\cdot}34737$.

Substituting (5.4) into (2.13), we find that the thickness function at the center of the plate, which is independent of the value of ν, is proportional to the square of the radius.

The result of the numerical solution of the optimization problem (3.1)–(3.3) and (5.1a)–(5.1d) is illustrated in Fig. 4, and corresponding to $\nu = 0{\cdot}3$ we find the optimal eigenvalue to be

$$\lambda_1 = 12(1-\nu^2)\omega_1^2\rho R^8/EV^2 = 24{\cdot}74. \tag{5.8}$$

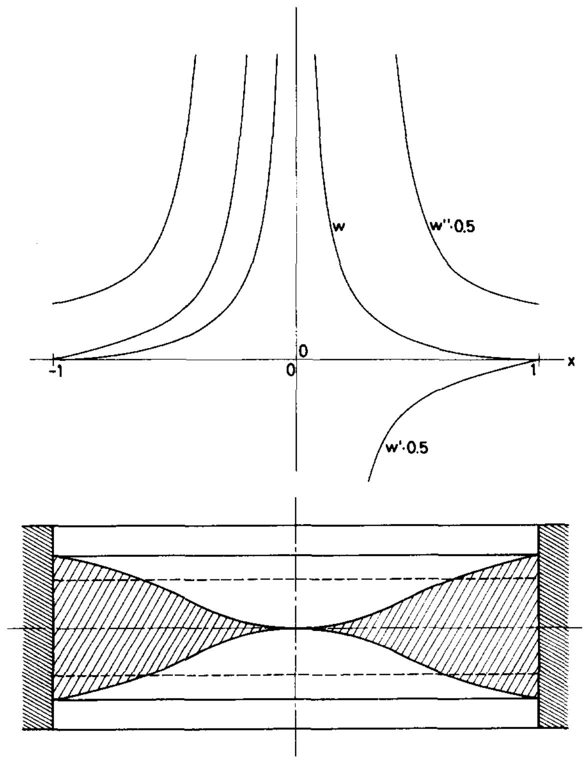

FIG. 4. The optimal shape of a clamped circular plate.

In this case, the first natural frequency ω_1 of the optimal plate is increased with about 53 pct. in comparison with the corresponding frequency of a uniform clamped plate of equal material, volume and diameter.

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Абстракт—В отношении к заданному объему определяется форма вращательно симметрической пластинки в такой форме, что ее первые собственные частоты поперечных колебаний становлятся оптимальными. Исследуются три разных сличая граничных условий.

Принимая, что найболее низкий вид колебаний вращательно симметрический, соответствующая математическая задача оказывается задачой на собственные значения четвертого порядка, нелинейной и сингулярной, но однородной. Выводится дифференциальное уравнение вариационным способом и решается методом последовательной итерации.